Anti-Symmetrically Fused Model and Non-Linear Integral Equations in the Three-State Uimin-Sutherland Model

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Abstract

We derive the non-linear integral equations determining the free energy of the three-state pure bosonic Uimin-Sutherland model. In order to find a complete set of auxiliary functions, the anti-symmetric fusion procedure is utilized. We solve the non-linear integral equations numerically and see that the low-temperature behavior coincides with that predicted by conformal field theory. The magnetization and magnetic susceptibility are also calculated by means of the non-linear integral equation.

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1 Introduction

The Bethe ansatz (BA) method [1] is a most fundamental approach to the study of one and two-dimensional exactly solvable lattice models. In the BA strategy, the quantum transfer matrix (QTM) method has been found quite a powerful tool to investigate the thermodynamics of several spin models and electron systems [2]-[9]. In most cases, after applying the BA it proved useful to introduce *auxiliary functions* which are ratios of the components of the eigenvalue of the QTM and their non-linear integral equation (NLIE). The main merit to utilize the NLIE is that we do not have to use the string hypothesis [10], which sometimes raises fundamental questions about its validity as well as pragmatical questions about the accuracy of the unavoidable truncation procedures of the typically infinitely many integral equations in the traditional thermodynamical Bethe ansatz (TBA) [10].

Once discovering a good (finite) set of auxiliary functions, we can get as good an accuracy as we need, because no truncation is necessary. Several remarkable studies of the thermodynamics [3, 6, 7] and the mathematical aspects [8] have been achieved by this scheme.

In this paper, we consider the one-dimensional Uimin-Sutherland (US) model [11]. For m fermionic and n bosonic components the system is referred to as the (m, n) model. For this general case the quantum transfer matrix is constructed and its eigenvalue equations are derived. The $(3, 0)$-US model is studied in detail for which the *spins* attached to the sites take three states, which can be regarded as the three vectors of the vector representation of the Lie algebra $sl(3)$. We have succeeded in finding the NLIE for the $(3, 0)$-US model with regular analyticity properties. To obtain these well-posed NLIE for the $(3, 0)$-US model, it is crucial to consider the anti-symmetrical fusion (ASF) model in addition to the defining one. The ASF of the vector representation of $sl(3)$ is another three-dimensional one, and thus called *conjugate*. The spinon like picture in the $(3, 0)$-US model is made clear with the help of the NLIE. As an application some physical quantities like specific heats are calculated with good accuracy by numerical treatments of the NLIE.

The remainder of this paper is organized as follows. We give an explanation of the US model and the QTM method in Sec.2. This section is also intended to fix notations. In Sec.3, we apply the BA to the QTM of the general US model and we also introduce the NLIE for the (3,0) model. In several limiting cases, for example the $SU(2)$-limit, exact analytic calculations are performed. In Sec.4, we solve the NLIE numerically and obtain the entropy, specific heat, magnetization, and magnetic susceptibility. Some low-temperature properties exposed by these data are discussed. In Sec.5 we give a summary of our work.

2 The Uimin-Sutherland model and its QTM

Let us begin with the review and definition of the general US models. The general one-dimensional q-state US model is defined as follows. Consider a one-dimensional lattice with L sites and periodic boundary conditions imposed. A q-state spin variable α_i is assigned to each site i. We can generally consider the situation where each spin α has its own grading, i.e. statistics number $\epsilon_\alpha = (-1)^{\xi_\alpha} = \pm 1$. A spin α with $\epsilon_\alpha = +1$ ($\epsilon_\alpha = -1$) is called bosonic (fermionic). The Hamiltonian of the US model can be introduced as

$$\mathcal{H}_0 = \sum_{i=1}^{L} \pi_{i,i+1} \tag{1}$$

with the permutation operator $\pi_{i,i+1}$

$$\pi_{i.i+1}|\alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_L\rangle = (-1)^{\xi_{\alpha_i \alpha_{i+1}}} |\alpha_1 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_L\rangle,$$

where $\xi_{\alpha_i \alpha_{i+1}}$ is 1 if both α_i and α_{i+1} are fermionic, and 0 otherwise.

Model (1) is shown to be exactly solvable on the basis of the Yang-Baxter equation. Many well-known exactly solvable models are of type (1), e.g. the spin-1/2 Heisenberg chain corresponds to $q = 2$ and $\epsilon_1 = \epsilon_2 = +1$, the free fermion model to $q = 2$ and $\epsilon_1 = -\epsilon_2 = +1$, the supersymmetric t-Jmodel to $q = 3$ and $\epsilon_1 = -\epsilon_2 = \epsilon_3 = +1$. If m of q ϵ's are $+1$ and $n(= q - m)$ are -1, for example, $\epsilon_1 = \cdots = \epsilon_m = +1$, $\epsilon_{m+1} = \cdots \epsilon_q = -1$, we call the model (m, n)-US model.

Now we want to consider the (3,0)-US model ($q = 3$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = +1$) whose Hamiltonian is equivalent to that of the $SU(2)$ spin-1 chain defined by

$$\mathcal{H}\prime = \sum_{i=1}^{L} \left[\vec{S}_i \cdot \vec{S}_{i+1} + (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right].$$

For the study of the thermodynamics we introduce the QTM as follows. First, the classical counterpart to (1) the Perk-Schultz (PS) model [12] is defined on a two-dimensional square lattice of $L \times N$ sites, where we impose periodic boundary conditions throughout this paper. We assume that variables taking on values $1, 2, \cdots, q$ are assigned to the bonds of the lattice. The Boltzmann weight associated with a local vertex configuration α, β, μ and ν is denoted by $R^{\mu\nu}_{\alpha\beta}(v)$, where v is the spectral parameter (Fig.1).

Using the Yang-Baxter equation, a lattice model defined by

$$R^{\mu\nu}_{\alpha\beta}(v) = \delta_{\alpha\nu}\delta_{\mu\beta} + v \cdot (-1)^{\xi_\alpha \xi_\mu} \cdot \delta_{\alpha\beta}\delta_{\mu\nu} \tag{2}$$

is proved to be exactly solvable. Defining the row-to-row transfer matrix

$$\mathcal{T}^{\beta}_{\alpha}(v) = \sum_{\{\mu\}} \prod_{i=1}^{L} R^{\mu_i \mu_{i+1}}_{\alpha_i \beta_i}(v),$$

the partition function is given by $\mathsf{Z}_{L,N} = Tr \mathcal{T}^N(v)$ where the trace is taken in the q^L-dimensional space. In this paper we are not primarily interested in the PS-model itself, but rather in its Hamiltonian limit obtained from $\mathcal{T}$. Making use of Baxter's formula [13] at $v = 0$

$$\mathcal{H}_0 = \frac{d}{dv} \ln \mathcal{T}(v) \bigg|_{v=0} = \sum_{i=1}^{N} \pi_{i,i+1}, \tag{3}$$

we get the Hamiltonian of the q-state US model.

The main idea of the quantum transfer matrix (QTM) method at finite temperature is as simple as follows (for details the reader is referred to the papers [3]). First, let us define a new set of vertex weights $\bar{R}(v)$ by rotating $R(v)$ by 90 degrees as

$$\bar{R}^{\mu\nu}_{\alpha\beta}(v) = R^{\alpha\beta}_{\nu\mu}(v)$$

and consider the transfer-matrix $\bar{\mathcal{T}}$ which is the product of $\bar{R}(v)$. Upon introducing a large integer N (Trotter number), we get the following relation with arbitrary reciprocal

temperature β

$$\mathcal{T}(-\beta/N)\bar{\mathcal{T}}(-\beta/N) = e^{-\frac{2\beta}{N}\mathcal{H}+O((\beta/N)^2)}.$$

Finally, the partition function of the one-dimensional Hamiltonian $\mathcal{H}$ at finite temperature $T = 1/\beta$ can be calculated by means of the following "Trotter-Suzuki" formula

$$\mathsf{Z} = \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \lim_{N\to\infty} \left(\mathrm{Tr}\, \mathcal{T}(u)\bar{\mathcal{T}}(u)\right)^{N/2}, \quad u = -\beta/N. \tag{4}$$

In other words, the finite-temperature partition function for the one-dimensional Hamiltonian is calculated as that of a *staggered* two-dimensional vertex model. For technical reasons, it is convenient to define another vertex weight $\tilde{R}(v)$ as a rotation of $R(v)$ by -90 degrees

$$\tilde{R}^{\mu\nu}_{\alpha\beta}(v) = R^{\beta\alpha}_{\mu\nu}(-v).$$

With these preparations, the quantum transfer matrix $\mathcal{T}_{\mathrm{QTM}}$ corresponding to the contribution of columns to the partition function is given by

$$\left(\mathcal{T}^{\mathrm{QTM}}\right)^{\beta}_{\alpha}(v) = \sum_{\mu}\prod_{j=1}^{N/2} R^{\mu_{2j-1}\mu_{2j}}_{\alpha_{2j-1}\beta_{2j-1}}(iv+u)\tilde{R}^{\mu_{2j}\mu_{2j+1}}_{\alpha_{2j}\beta_{2j}}(iv-u). \tag{5}$$

Here we have introduced a spectral parameter v such that $\mathcal{T}_{\mathrm{QTM}}(v)$ is a commuting family of matrices generated by v. This will allow us to diagonalize $\mathcal{T}_{\mathrm{QTM}}$. The final results, of course, are physically interesting only for $v = 0$ as the partition function of the one-dimensional US model at temperature $1/\beta$ is given by

$$\mathcal{Z} = \lim_{N\to\infty} \mathrm{Tr}\, \mathcal{T}^{L}_{\mathrm{QTM}}(0).$$

The free energy per unit length is

$$f = -\lim_{L\to\infty}\frac{1}{L\beta}\ln \mathsf{Z} = -\frac{1}{\beta}\ln\Lambda_{\mathrm{max}}(0),$$

where $\Lambda_{\mathrm{max}}(v)$ is the largest eigenvalue of the QTM. If we succeed in obtaining the next-largest eigenvalue $\Lambda_1(v)$, the correlation length ξ at the finite temperature $T = 1/\beta$ is given by

$$\xi^{-1} = -\lim_{N\to\infty}\ln\left|\frac{\Lambda_1}{\Lambda_{\mathrm{max}}}\right|.$$

3 Nonlinear Integral Equation for the Free Energy

Let us apply the BA for this QTM. While keeping solvability, we can add external field terms $\mathcal{H}_{\rm ext}$ to $\mathcal{H}_0$ like

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm ext} = \mathcal{H}_0 - \sum_{i=1}^{L} \sum_{\alpha=1}^{q} \mu_\alpha n_{i,\alpha}.$$

The BA for the eigenvalues of the QTM of the general q-state PS model is conjectured to take the form of

$$\begin{aligned}
\Lambda(v) &= \sum_{j=1}^{q} \lambda_j(v), \\
\lambda_1(v) &= \prod_{k_1=1}^{M_1} \frac{v - v^1_{k_1} + i\epsilon_1}{v - v^1_{k_1}} (v - iu - i\epsilon_1)^{\frac{N}{2}} (v + iu)^{\frac{N}{2}} e^{\beta\mu_1}, \\
\lambda_j(v) &= \prod_{k_{j-1}=1}^{M_{j-1}} \frac{v - v^{j-1}_{k_{j-1}} - i\epsilon_j}{v - v^{j-1}_{k_{j-1}}} \prod_{k_j=1}^{M_j} \frac{v - v^j_{k_j} + i\epsilon_j}{v - v^j_{k_j}} (v - iu)^{\frac{N}{2}} (v + iu)^{\frac{N}{2}} e^{\beta\mu_j}, \\
&\qquad\qquad (j = 2, \cdots, q-1), \\
\lambda_q(v) &= \prod_{k_{q-1}=1}^{M_{q-1}} \frac{v - v^{q-1}_{k_{q-1}} - i\epsilon_q}{v - v^{q-1}_{k_{q-1}}} (v + iu + i\epsilon_q)^{\frac{N}{2}} (v - iu)^{\frac{N}{2}} e^{\beta\mu_q}.
\end{aligned} \tag{6}$$

We concentrate on the $(3,0)$-US model. Defining $\mathsf{q}_i(v) = \prod(v - v^i_{k_i})$ and $\phi_\pm(v) = (v \pm iu)^{N/2}$, the last formulas are reduced to

$$\begin{aligned}
\lambda_1(v) &= \frac{\mathsf{q}_1(v+i)}{\mathsf{q}_1(v)} \phi_+(v)\phi_-(v-i) e^{\beta\mu_1}, \\
\lambda_2(v) &= \frac{\mathsf{q}_1(v-i)}{\mathsf{q}_1(v)} \frac{\mathsf{q}_2(v+i)}{\mathsf{q}_2(v)} \phi_+(v)\phi_-(v) e^{\beta\mu_2}, \\
\lambda_3(v) &= \frac{\mathsf{q}_2(v-i)}{\mathsf{q}_2(v)} \phi_+(v+i)\phi_-(v) e^{\beta\mu_3}, \\
\Lambda(v) &= \lambda_1(v) + \lambda_2(v) + \lambda_3(v).
\end{aligned} \tag{7}$$

We have checked numerically for small N's that the above BA gives the correct eigenvalues.

Next, we want to define auxiliary functions which determine the (largest) eigenvalue completely and satisfy a closed system of integral equations. In the case of the t-Jmodel, a complete set of auxiliary functions is found [6] to be

$$b = \frac{\lambda_1}{\lambda_2 + \lambda_3}, \; \bar{b} = \frac{\lambda_3}{\lambda_1 + \lambda_2}, c = \frac{\lambda_1 \lambda_3}{\lambda_2(\lambda_1 + \lambda_2 + \lambda_3)}. \tag{8}$$

Unfortunately, for the $(3,0)$-US model the above set does not admit a closed set of functional equations to determine the eigenvalue. In order to complete this set we found in the fusion procedure [14] a useful working basis [7, 9]. The fusion model is defined as follows. The PS model given by (2) is identified with the vector representation of the $sl(m|n)$ model. On the basis of tensor products and proper projections, we get the other representations with higher levels. The solvability of the fusion model is again guaranteed by the Yang-Baxter relation. Here we consider only the ASF model. For the case of $sl(n)$, we call the $(n-1)$-th ASF model the *conjugate model* because the ASF of the fundamental model and the $(n-1)$-th ASF is zero-dimensional. This conjugacy is somewhat like that between quark and anti-quark representations of $SU(3)$. For the $sl(2)$ case, the ASF of two vector representations (*i.e* spin-1/2) gives the zero-dimensional representation (*i.e* spin-0). Therefore, we call the $sl(2)$ PS model self-conjugate. Under the conjugate transformation, we see

$$\boxed{1} \leftrightarrow \boxed{2} \quad .$$

For the $sl(3)$ case,

$$\boxed{1} \leftrightarrow \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\ \boxed{2} \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,,\ \boxed{3} \leftrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,.$$

The BA equation for the fusion model is obtained by the simple replacement

$$\lambda_1(v) \to \lambda_{2,3}(v),\ \lambda_2(v) \ \to \ \lambda_{1,3}(v),\ \lambda_3(v) \to \lambda_{1,2}(v), \tag{9}$$

$$\Lambda(v) \to \tilde{\Lambda}(v) \ = \ \lambda_{2,3}(v) + \lambda_{1,3}(v) + \lambda_{1,2}(v), \tag{10}$$

where $\lambda_{l,m}(v)$ is defined by

$$\lambda_{l,m}(v) = \lambda_l(v)\lambda_m(v+i).$$

We are ready to define the auxiliary functions. In the case of the $sl(2)$ PS model, the ASF model is nothing but the fundamental model itself. As is known, a complete set of auxiliary functions consists of $p = \lambda_1/\lambda_2$ and $\bar{p} = p^{-1} = \lambda_2/\lambda_1$. In fact, p and $\bar{p}$ are conjugates. For the $sl(3)$ case, the three functions b, $\bar{b}$ and c are not complete in the $(3,0)$-US model, however the 6 functions b, $\bar{b}$, c and their conjugates constitute a complete

set. For this reason we have introduced the conjugate transformation. Let us proceed this program. In dependence on the real variable x, we define the functions

$$
\begin{aligned}
s_1(x) &= \left.\frac{\lambda_1}{\lambda_2+\lambda_3}\right|_{v=x+i/2} \quad , \quad s_2(x) = \left.\frac{\lambda_{12}\lambda_{23}}{\lambda_{13}(\lambda_{12}+\lambda_{23}+\lambda_{13})}\right|_{v=x-i/2} , \\
s_3(x) &= \left.\frac{\lambda_3}{\lambda_1+\lambda_2}\right|_{v=x-i/2} \quad , \quad s_4(x) = \left.\frac{\lambda_{12}}{\lambda_{13}+\lambda_{23}}\right|_{v=x} , \\
s_5(x) &= \left.\frac{\lambda_1\lambda_3}{\lambda_2(\lambda_1+\lambda_2+\lambda_3)}\right|_{v=x} \quad , \quad s_6(x) = \left.\frac{\lambda_{23}}{\lambda_{12}+\lambda_{13}}\right|_{v=x-i} , \\
\Lambda(x) &= \lambda_1+\lambda_2+\lambda_3|_{v=x} \quad , \quad \bar{\Lambda}(x) = \lambda_{12}+\lambda_{23}+\lambda_{13}|_{v=x-i/2} .
\end{aligned} \tag{11}
$$

After a lengthy calculation using the Fourier transform the following non-linear integral equation are proved explicitly

$$
\begin{pmatrix} \ln s_1(x) \\ \ln s_2(x) \\ \ln s_3(x) \\ \ln s_4(x) \\ \ln s_5(x) \\ \ln s_6(x) \end{pmatrix} = - \begin{pmatrix} \beta\epsilon_1(x) \\ \beta\epsilon_2(x) \\ \beta\epsilon_3(x) \\ \beta\epsilon_4(x) \\ \beta\epsilon_5(x) \\ \beta\epsilon_6(x) \end{pmatrix} + \begin{pmatrix} K_0 & -K_1 & -K_1 & -K_3 & -K_3 & -K_4 \\ -K_2 & K_0 & -K_1 & -K_3 & -K_6 & -K_3 \\ -K_2 & -K_2 & K_0 & -K_5 & -K_3 & -K_3 \\ -K_3 & -K_3 & -K_4 & K_0 & -K_1 & -K_1 \\ -K_3 & -K_6 & -K_3 & -K_2 & K_0 & -K_1 \\ -K_5 & -K_3 & -K_3 & -K_2 & -K_2 & K_0 \end{pmatrix} * \begin{pmatrix} \ln S_1(x) \\ \ln S_2(x) \\ \ln S_3(x) \\ \ln S_4(x) \\ \ln S_5(x) \\ \ln S_6(x) \end{pmatrix} . \tag{12}
$$

Several remarks are in order. First, $S_i = 1 + s_i \quad (i = 1, \cdots, 6)$. Second, the driving terms/bare energies of the *spinons* are defined as

$$\epsilon_1(x) = V_1(x) + (-2\mu_1 + \mu_2 + \mu_3)/3, \tag{13}$$

$$\epsilon_2(x) = V_1(x) + (\mu_1 - 2\mu_2 + \mu_3)/3, \tag{14}$$

$$\epsilon_3(x) = V_1(x) + (\mu_1 + \mu_2 - 2\mu_3)/3, \tag{15}$$

$$\epsilon_4(x) = V_2(x) + (-\mu_1 - \mu_2 + 2\mu_3)/3, \tag{16}$$

$$\epsilon_5(x) = V_2(x) + (-\mu_1 + 2\mu_2 - \mu_3)/3, \tag{17}$$

$$\epsilon_6(x) = V_2(x) + (2\mu_1 - \mu_2 - \mu_3)/3 \tag{18}$$

with

$$V_1(x) = \frac{2\pi}{\sqrt{3}} \frac{1}{2\cosh(2\pi x/3) - 1}, \quad V_2(x) = \frac{2\pi}{\sqrt{3}} \frac{1}{2\cosh(2\pi x/3) + 1}. \tag{19}$$

Lastly, the kernels $K_l(x)(l = 0, \cdots, 6)$ are

$$K_l(x) = \int_{-\infty}^{\infty} dx e^{ikx} K_l(k)$$

with

$$
\begin{aligned}
K_0(k) &= \frac{e^{-|k|}}{e^k+1+e^{-k}},\ K_1(k) = \frac{1+e^{-3k/2-|k|/2}}{e^k+1+e^{-k}},\ K_2(k) = \frac{1+e^{3k/2-|k|/2}}{e^k+1+e^{-k}}, \\
K_3(k) &= \frac{e^{|k|/2}}{e^k+1+e^{-k}},\ K_4(k) = \frac{e^{-3k/2-|k|}}{e^k+1+e^{-k}}, K_5(k) = \frac{e^{3k/2-|k|}}{e^k+1+e^{-k}}, \\
K_6(k) &= \frac{e^{-|k|/2}+2e^{|k|/2}+e^{-3|k|/2}}{e^k+1+e^{-k}},
\end{aligned} \tag{20}
$$

and the convolution is defined by

$$
f * g(x) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} f(x-y)g(y).
$$

We introduce the rescaled eigenvalue $\Lambda'(v) = \Lambda(v)/[\phi_+(v+i)\phi_-(v-i)]$ which is useful, because of the simple asymptotic behaviour $\lim_{v\to\infty} \Lambda'(v) = \text{const}$. At $v = 0$, the relation of eigenvalue and rescaled eigenvalue is simply

$$
\ln \Lambda(0) = \ln \Lambda'(0) - \beta,
$$

i.e. it only amounts to a shift in the ground state energy. In terms of the auxiliary functions the rescaled eigenvalue reads

$$
\begin{aligned}
\ln \Lambda'(x) = -\beta e(x) \quad &+ \quad V_1 * \ln S_1(x) + V_1 * \ln S_2(x) + V_1 * \ln S_3(x) \\
&+ \quad V_2 * \ln S_4(x) + V_2 * \ln S_5(x) + V_2 * \ln S_6(x)
\end{aligned}
$$

with

$$
e(0) = -\int dk \frac{1+e^{-|k|}}{e^k+1+e^{-k}} = -\left(\frac{\pi}{3\sqrt{3}} + \ln 3\right).
$$

From considering the low temperature limit by following [3], we conclude that there are two kinds of elementary excitations. One of them corresponds to the vector representation (s_1, s_3, s_5), and the other to its conjugate (s_2, s_4, s_6). Their bare energies are represented by $\epsilon_1, \cdots \epsilon_6$.

Limiting cases

Next, we take a suitable limit of the chemical potentials μ_1, μ_2 and μ_3, for which the spin-1/2 Heisenberg model is obtained. It is quite a natural result on the level of the Hamiltonian formalism, however less trivial on the level of the final NLIE, which supports the validity of our construction and the correctness of our calculations. Let us concentrate on the case of $\mu_1 = \mu' + h_1$, $\mu_3 = \mu' + h_3$ and $\mu_2 = h_2$, where $|h_1|, |h_2|, |h_3| \ll |\mu'|$. First consider the limit $\mu' = \infty$. In this case, observing the constants in the driving terms of each auxiliary function, we conclude

$$s_1 = O(1), \quad s_2 = O(e^{-2\beta\mu'}), \quad s_3 = O(1), \tag{21}$$

$$s_4 = O(e^{-\beta\mu'}), \quad s_5 = O(e^{\beta\mu'}), \quad s_6 = O(e^{-\beta\mu'}). \tag{22}$$

Therefore, we can regard

$$\begin{aligned} s_2 = s_4 &= s_6 = 0, \\ s_5 &\sim S_5 \end{aligned} \tag{23}$$

that means the conjugate modes are suppressed. With the above ansatz, we can solve the equation for the S_2-function as

$$\ln S_5(x) = K_3 * (K_0 - 1)^{-1} * (\ln S_1(x) + \ln S_3(x)) + \beta(K_0 - 1)^{-1} * V_2(x)$$

Substituting the last equation into (12) for s_1 and s_3 and performing the inverse Fourier transform, we get

$$\begin{aligned} \ln p(x + i/2) &= -2\pi\beta\Phi(x + i/2) + \beta h/2 \\ &\quad + \int_{-\infty}^{\infty} \frac{dy}{2\pi} k(x - y) \ln P(y + i/2) - \int_{-\infty}^{\infty} \frac{dy}{2\pi} k(x - y + i) \ln \bar{P}(y - i/2), \\ \ln \bar{p}(x - i/2) &= 2\pi\beta\Phi(x - i/2) - \beta h/2 \\ &\quad - \int_{-\infty}^{\infty} \frac{dy}{2\pi} k(x - y - i) \ln P(y + i/2) + \int_{-\infty}^{\infty} \frac{dy}{2\pi} k(x - y) \ln \bar{P}(y - i/2), \\ \ln \Lambda(x) &= 2\beta \ln 2 + \beta\mu + \frac{i}{2} \int_{-\infty}^{\infty} dx \frac{\ln P(x + i/2)}{\sinh \pi(x + i/2)} - \frac{i}{2} \int_{-\infty}^{\infty} dx \frac{\ln \bar{P}(x - i/2)}{\sinh \pi(x - i/2)} \end{aligned} \tag{24}$$

with

$$p(x+i/2)=s_1(x),\ \bar{p}(x-i/2)=s_3(x),\ P(v)=1+p(v),\ \bar{P}(v)=1+\bar{p}(v),$$
$$\Phi(v)=-\frac{i}{2}\frac{1}{\sinh\pi v},\ k(x)=\int\frac{dx}{2\pi}\frac{e^{ikx}}{1+e^{|k|}},\ h=h_1-h_3,\ \mu=\frac{h_1+h_2+h_3}{3}, \quad (25)$$

where we have dropped the contribution of μ' to the potential μ. This NLIE is nothing but that for the spin 1/2-Heisenberg model [3]. Thus, we see that the spin 1/2-Heisenberg model is obtained as a limit of the $(3,0)$-US model directly in the two sets of NLIE's.

Now, let us consider the opposite limit $\mu'=-\infty$. In this case, we find the following simplifications

$$\begin{aligned} s_1=s_3 &= s_5=0, \\ s_2 &\sim S_2. \end{aligned} \quad (26)$$

Therefore, we obtain

$$\ln S_2(x)=K_3*(K_0-1)^{-1}*(\ln S_4(x)+\ln S_6(x))+\beta(K_0-1)^{-1}*V_1(x). \quad (27)$$

However, substituting (27) into (12), we see that s_4, s_6 and Λ are constants. This fact is reasonable, because in the present case only the 2nd state can survive with finite energy. Hence, all physical degrees of freedom are frozen out at finite temperature.

4 Numerical Analysis of the NLIE

As an application of the obtained NLIE (12), we show numerical results for some physical quantities. Let us consider the entropy and the specific heat:

$$S=-\left(\frac{\partial f}{\partial T}\right),\quad C=T\left(\frac{\partial S}{\partial T}\right). \quad (28)$$

To avoid numerical differentiations, we simultaneously have solved the NLIE for the derivatives using relations like [6]

$$\frac{\partial}{\partial\beta}\ln(1+s_i)=\frac{s_i}{1+s_i}\frac{\partial}{\partial\beta}\ln s_i.$$

We have calculated the entropy and specific heat of the $(3,0)$-US model with $\mu_1=\mu_2=\mu_3=0$ numerically as shown in Fig.2.

We can compare the result with that from the $SL(3)_1$ conformal field theory (CFT). From CFT and its finite size analysis, the free energy is predicted to be

$$f = f_0 - \frac{\pi c}{6v} T^2 + \cdots . \tag{29}$$

Putting $c = 2$ and $v = 2\pi/3$ [16], the low-temperature asymptotics is

$$C = T.$$

With a glance at Fig.2, we see that the low temperature behaviour of the numerically determined specific heat has slope 1 as predicted by CFT. In a similar manner, we can calculate the specific heat in the presence of the chemical potential. First, we show results for the case $\mu_1 = \mu_3 = \mu$, $\mu_2 = 0$ in Figs.3a and 3b. As we have analyzed before, the specific heat in this case approaches that of the $SU(2)$ spin-1/2 Heisenberg chain with zero magnetic field as $\mu \to \infty$. For the Heisenberg chain, we should put $c = 1$ and $v = \pi$ and get the asymptotics $C = T/3$. Indeed, the curves in Fig.3a show the expected tendency. Results for the specific heat for $\mu_1 = \mu_3 = \mu < 0$, $\mu_2 = 0$ are shown in Fig.3b. Here we see a suppression of the specific heat data at fixed temperature for $\mu \to -\infty$. This is the manifestation of the freezing of the system. Second, the case of $\mu_1 = -\mu_3 = h$, $\mu_2 = 0$ is considered in Fig.4. In this case, the rich $sl(3)$ structure is most clearly exposed.

At $h = 4$ in Fig.4, we observe two structures (one shoulder and one maximum) of the specific heat owing to the fundamental mode and its conjugate.

Let us define the magnetization M and the magnetic susceptibility χ by

$$M = -\frac{\partial f}{\partial h}, \quad \chi = \left.\frac{\partial M}{\partial h}\right|_{h=0} = -\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0},$$

respectively. The results are shown in Figs. 5a, b and 6. From Figs.5 we read off the existence of two critical field $h_{c1} \cong 0.94$ and $h_{c2} = 4$. For $h \geq h_{c2}$ only the $\alpha = 1$ state can survive at $T = 0$, thus the groundstate magnetization is maximal, i.e. $M = 1$. This is in agreement with analytic considerations. The lower field h_{c1} is the value, above which only the $\alpha = 1$ and $\alpha = 2$ states can exist. Note that for these critical fields the magnetization $M(T)$ shows a square root behaviour at low T.

The magnetic susceptibility is given in Fig.6. The curve is qualitatively similar to that of $SU(2)$ Heisenberg model however with $\chi(0) = 3/\pi^2$. The susceptibilities at the lowest temperatures shown in Fig. 6 are still about 10% above the groundstate value. The origin of this singular behaviour of the susceptibility at $T = 0$ are $1/\log T$ corrections similar to [17, 18] and will be discussed elsewhere.

5 Summary

Before closing this paper, we would like to mention another application and generalizations of the NLIE. First, we can use our NLIE in order to calculate the characters of the $SL(3)_1$ Kac-Moody algebra. Let us remember that our NLIE contains a natural notion of spinons and chemical potentials [19]-[21]. This fact enables us to present the characters in terms of the spinons and chemical potentials [22]. Details will be published elsewhere. Second, up to our knowledge the complete strings in the anisotropic $sl(n)$ ($n > 2$) models with trigonometric and elliptic R-matrices have not been constructed yet. However, one of the present authors has shown that the QTM method is applicable not only to the isotopic XXX Heisenberg chain but also to the anisotropic XXZ and XYZ versions [3]. Therefore, we expect that a combination of the anti-symmetric fusion procedure and the NLIE will be generalizable to the anisotropic $sl(3)$ models.

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Figure Captions

Fig.1 R-matrix for the PS model.

Fig.2 Entropy and specific heat in the absence of the chemical potentials.

Fig.3a Specific heat for $\mu_1 = \mu_3 = \mu \geq 0$.

Fig.3b Specific heat for $\mu_1 = \mu_3 = \mu \leq 0$.

Fig.4 Specific heat with $\mu_1 = -\mu_3 = h \geq 0$.

Fig.5a Magnetization for $\mu_1 = -\mu_3 \cong h_{c2}$, $\mu_2 = 0$.

Fig.5b Magnetization for $\mu_1 = -\mu_3 \cong h_{c1}$, $\mu_2 = 0$.

Fig.6 Magnetic susceptibility.

$$R^{\mu\nu}_{\alpha\beta}(v) =$$

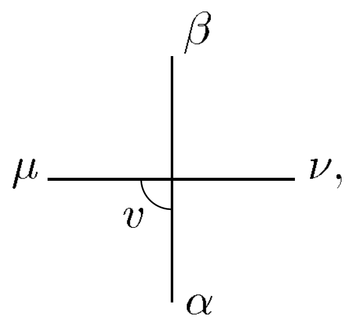


Fig.1

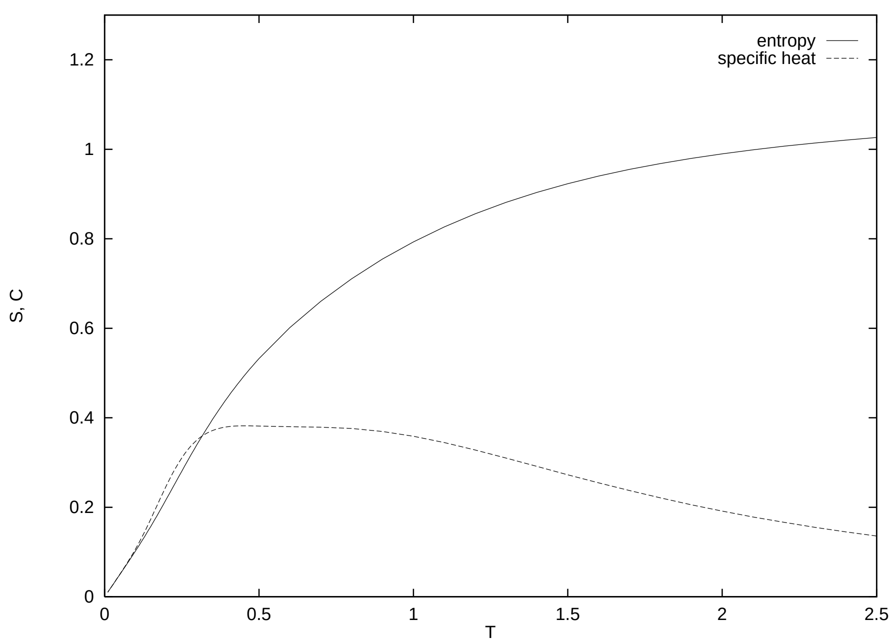


Fig.2

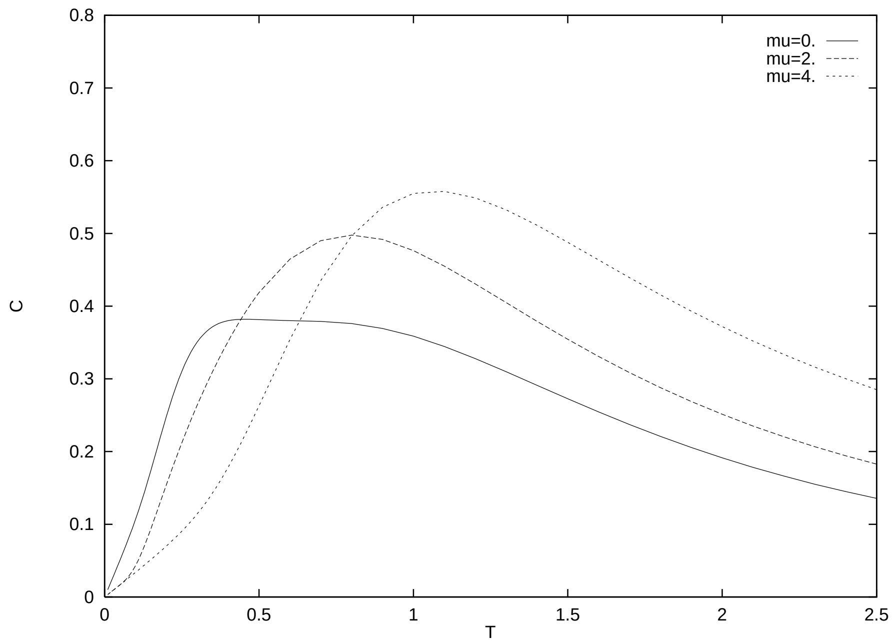

mu=0.
mu=2.
mu=4.
C
T
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
0.5
1
1.5
2
2.5

Fig.3a

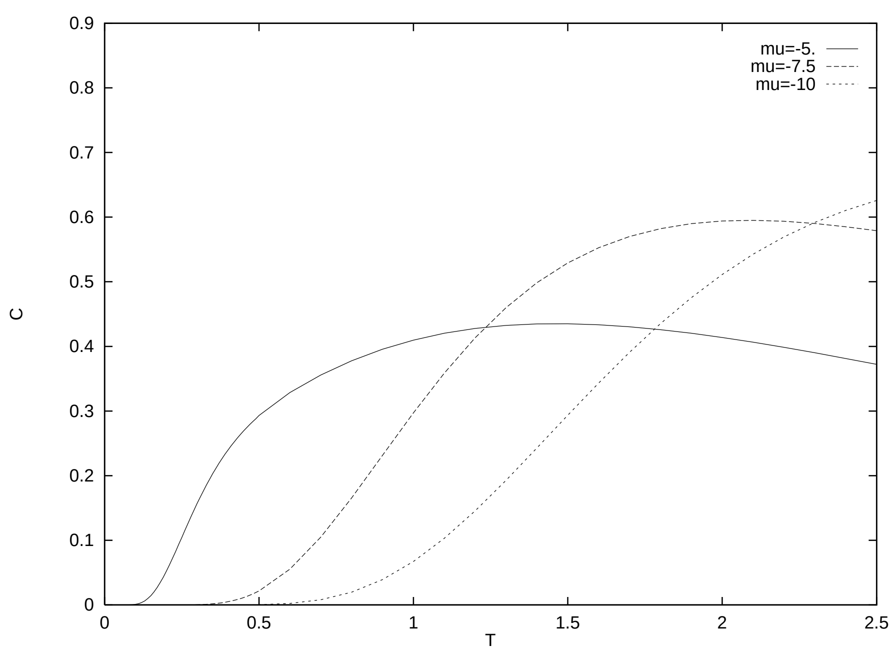
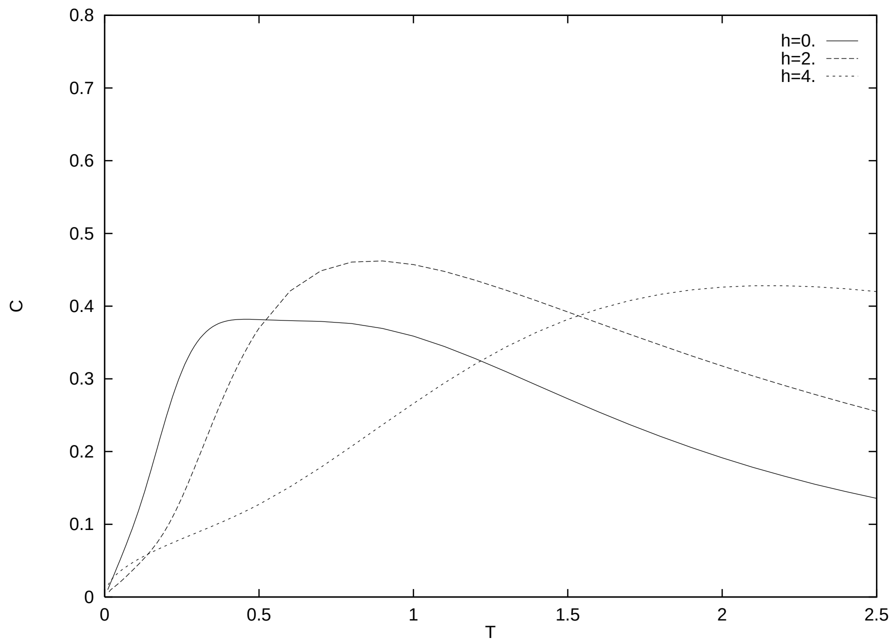
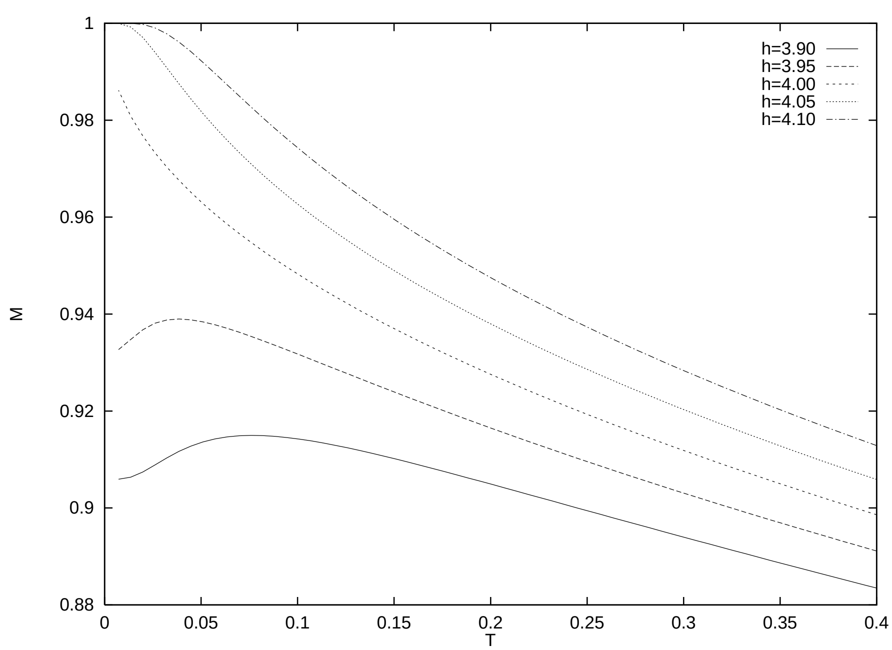

mu=-5.
mu=-7.5
mu=-10
C
T
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
0.5
1
1.5
2
2.5

Fig.3b



C
T
h=0.
h=2.
h=4.
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
0.5
1
1.5
2
2.5

Fig.4



M
T
h=3.90
h=3.95
h=4.00
h=4.05
h=4.10
0.88
0.9
0.92
0.94
0.96
0.98
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4

Fig.5a

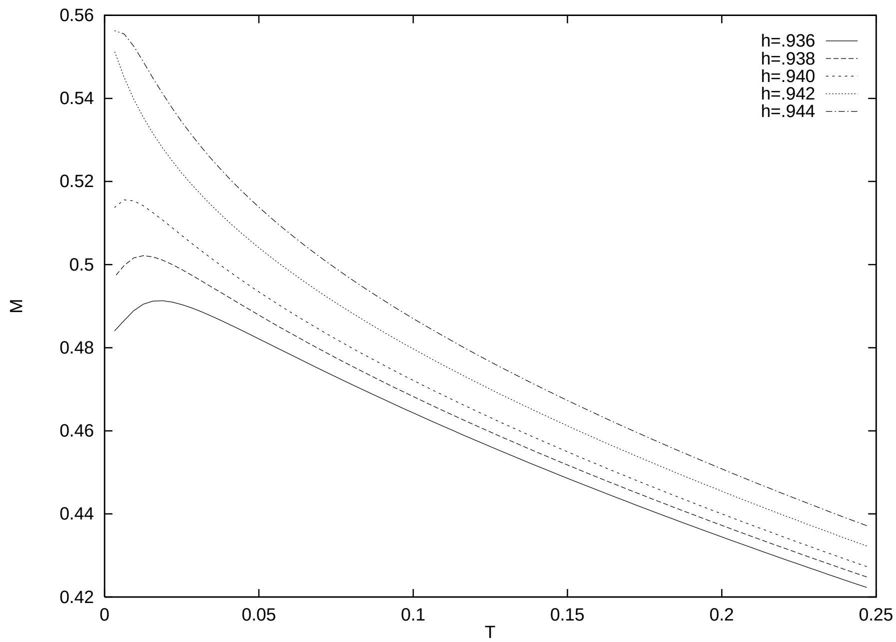

h=.936
h=.938
h=.940
h=.942
h=.944
M
T
0.56
0.54
0.52
0.5
0.48
0.46
0.44
0.42
0
0.05
0.1
0.15
0.2
0.25

Fig.5b

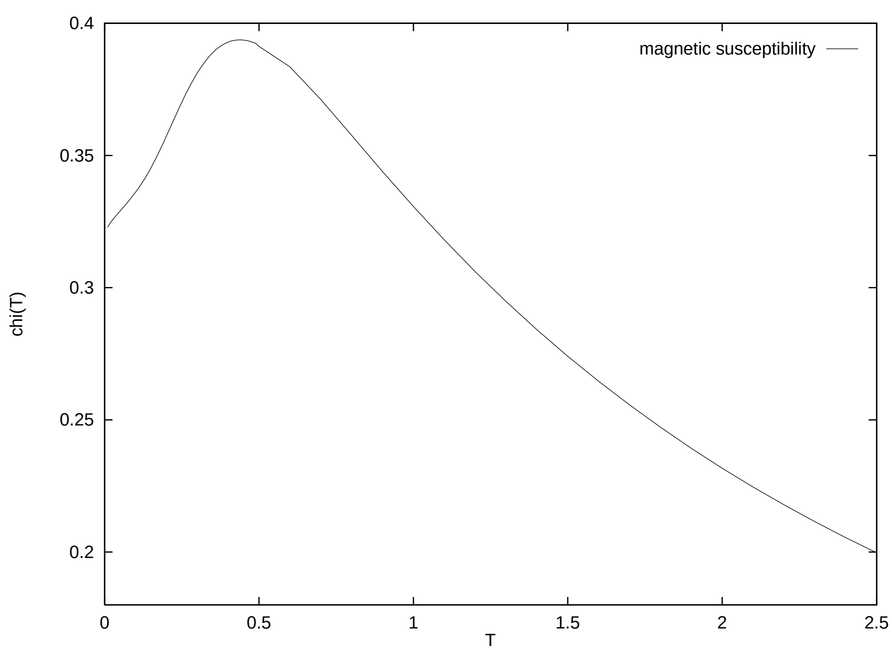

magnetic susceptibility
chi(T)
T
0.4
0.35
0.3
0.25
0.2
0
0.5
1
1.5
2
2.5

Fig.6